SKYRMIONS FROM SU(3) HARMONIC MAPS AND THEIR QUANTIZATION

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Static properties of $SU(3)$ multiskyrmions with baryon numbers up to 6 are estimated. The calculations are based on the recently suggested generalization of the $SU(2)$ rational map ansätze applied to the $SU(3)$ model. Both $SU(2)$ embedded skyrmions and genuine $SU(3)$ solutions are considered and it is shown that although, at the classical level, the energy of the embeddings is lower, the quantum corrections can alter these conclusions. This correction to the energy of the lowest state, depending on the Wess-Zumino term, is presented in the most general case.

1. Topological soliton models, and the Skyrme model among them [1], have recently generated a fair amount of interest because they may be able to describe various properties of low energy baryons. So far, most of such studies have involved the $SU(2)$ Skyrme model, whose solutions were then embedded into the $SU(3)$ model. This is justified as the solutions of the $SU(2)$ model are also solutions of $SU(N)$ models. However, there exist solutions of the $SU(N)$ models which are **not** embeddings of $SU(2)$ fields and it is important to assess their contribution.

As with solutions of the $SU(2)$ Skyrme models, the solutions of the $SU(3)$ model, with very few exceptions, can only be determined numerically. Like for the $SU(2)$ case [2] one starts with a harmonic map ansatz which gives fields “close” to the genuine solutions [3, 4]. Then one can use these fields as starting configurations of various numerical minimization schemes. For the $SU(2)$ fields such an approach was carried out in [2] where it reproduced the results of the earlier numerical approaches [5, 6] and so showed that the harmonic map approximations are very close to the final fields. The real solutions have energies only few hundreds of MeV lower than the harmonic approximations and the baryonic charge and energy distributions do not look very different *etc.* This has justified the use of harmonic field approximants in calculating various quantities when estimating quantum corrections to the classical results.

In this paper we take a look at the nonembedding solutions of the $SU(3)$ model. First we recall the results of [3] and use them in a $SU(3)$ numerical minimization program to estimate the values of energies of true solutions. The $SU(3)$ variational minimization program [7] has been rearranged for this purpose to allow the consideration of quite general field ansätze. This is discussed in the next section.

In the following section we discuss various quantum corrections and compare our results with the similar results for the embeddings. The paper ends with a short section presenting our conclusions.

2. The Lagrangian of the $SU(3)$ Skyrme model, in its well known form, depends on the parameters F_π and e and is given by [8, 9]:

$$\mathcal{L} = \frac{F_\pi^2}{16} Tr l_\mu l^\mu + \frac{1}{32e^2} Tr[l_\mu, l_\nu]^2 + \frac{F_\pi^2 m_\pi^2}{16} Tr(U + U^\dagger - 2) \tag{1}$$

Here $U \in SU(3)$ is a unitary matrix describing the chiral (meson) fields, and $l_\mu = U^\dagger \partial_\mu U$. In the model F_π should be fixed at the physical value: $F_\pi = 186$ Mev . The flavour symmetry breaking (FSB) in the Lagrangian will be considered below.

The Wess-Zumino term, which should to be added to the action, is given by

$$S^{WZ} = \frac{-iN_c}{240\pi^2} \int_\Omega d^5x \epsilon^{\mu\nu\lambda\rho\sigma} Tr(\tilde{l}_\mu \tilde{l}_\nu \tilde{l}_\lambda \tilde{l}_\rho \tilde{l}_\sigma), \tag{2}$$

where Ω is a 5-dimensional region with the 4-dimensional space-time as its boundary and where $\tilde{l}_\mu$ is a 5-dimensional analogue of $l_\mu = U^\dagger \partial_\mu U$. As is well known, this extra term does not contribute to the static masses of classical configurations, but it defines important topological properties of skyrmions [8, 9] and plays an important role in their quantization [10, 11].

In [3] an ansatz was presented which allows us to find approximate solutions of the $SU(3)$ model. This ansatz involves parametrising the static field as

$$U(\vec{x}) \;=\; \exp\left\{ f(r) \left(P(\theta,\varphi) - \frac{1}{3} \right) \right\}, \tag{3}$$

where r, θ and φ are polar coordinates, $f(r)$ is a radial profile function which has to be determined numerically and $P(\theta,\varphi)$ is a projector involved in the harmonic map ansatz. As shown in [3] this projector is given by

$$P \;=\; \frac{FF^\dagger}{|F|^2}, \tag{4}$$

where F is a 3-component vector, whose entries are polynomials in $z = tan(\frac{\theta}{2})e^{i\varphi}$. The largest degree of the polynomial gives the baryon number B of the final $SU(3)$ Skyrme field configuration. For an $SU(2)$ embedding the approach is similar except that this time we put

$$U(\vec{x}) = \begin{pmatrix} U_2 & \vec{0} \\ \vec{0} & 1 \end{pmatrix}, \tag{5}$$

where U_2 is an $U(2)$ matrix determined in an analogous way as (3) except that the vector F has only two components.

The fields F in Eq. (4) are so chosen that the configuration (3) has the smallest energy; this provides us with the harmonic map approximation to the real minimal energy static solution of the Skyrme model. We have then taken the expressions for F determined in [3] and used the corresponding U in (3) as an initial field in our minimization program. For $B = 2$ $F = (1, \sqrt{2}z, z^2)^T$, for $B = 3$ $F = (1/\sqrt{2}, 1.576z, z^3)^T$, for $B = 4$ $F = (1, 2.72z^2, z^4)^T$, for $B = 5$ $F = (4.5z^2, 2z^4 + 1, z^5 - 2.7z)^T$ and for $B = 6$ $F = (kz^3, 1 - 3z^5, z^6 + 3z)^T$ with $k = 7.06$ [3].

We have performed many minimizations of the field configurations involving baryon numbers $3, 4, 5$ and 6. In all cases the initial $SU(3)$ harmonic map fields had energies somewhat higher than the corresponding harmonic map embeddings.

Our $SU(3)$ minimization program, which uses the parametrization of the $SU(3)$ field in terms of two, mutually orthogonal, complex unit vectors[1] was then used to minimize the energy further. The constraints of the orthogonality and of the vectors being of unit length were replaced by the introduction of extra positive contributions to the energy (with large coefficients) - the so called "penalty terms".

The program itself minimized the energy using a mixture of a finite element and finite step methods, and we varied the coefficients of the penalty terms so as not to be trapped in a local minimum. In each case the final energy of the field configuration was

[1] The authors thank W.K. Baskerville for making them aware of this parametrisation

lower by a few hundreds of MeV from the energy of the harmonic approximant. However, due to the smallness of the lattice the baryon number was also somewhat lower; so our values in the **Table** were obtained by a linear extrapolation: $M = M + \Delta M$, where $\Delta M = \alpha * (B - B_{obs})$ and where B is the baryon number and B_{obs} is its value on the lattice. α was determined by looking at various minimizations (with different values of the coefficients of the penalty terms).

To check the stability of the program we also performed a series of minimizations when the initial field was a mixture of the $SU(3)$ harmonic map field and of the embedding. When the mixture was close to the embedding or to the $SU(3)$ harmonic map the minimization program took it down to these fields. When the mixture was far from either of these special fields it evolved to a new configuration of higher energy than either of the special ones. Although some of these new configurations can be numerical artifacts it is clear that the spectrum of static solutions of the $SU(3)$ model is very complicated, with most states having energies larger than the energies of the embeddings or the fields derived by the harmonic map ansatz. Hence our estimates in the **Table** have some physical justification.

In the **Table** we present some of our results. It is difficult to assess their accuracy; it is probably within 1% for the masses and several % for other quantities. In all numerical minimizations we have worked on a small tetrahedral lattice and hence we had a small "leakage" of both the energy and of the baryon number.

B	M_{cl}	$M_{cl}^{SU_2}$	$\Delta E_{M.t.}$	C_S	$\Theta_{1,2}$	Θ_3	Θ_4	Θ_5	$\Theta_{6,7}$	Θ_8	Θ_{38}	WZ_3	WZ_8	ΔE
1	1.70	1.70	46	—	5.56	5.56	2.04	2.04	2.04	–	–	-	-	368
2	3.38	3.26	89	0.33	5.70	6.40	7.10	7.10	5.70	5.0	-1.2	0.00	0.00	0.0
3	4.85	4.80	110	0.33	7.15	7.90	8.60	8.60	9.20	5.1	-1.9	-0.7	-0.15	35
4	6.48	6.20	174	0.31	12.2	9.80	11.6	11.6	12.1	6.2	-3.0	-1.0	0.6	80
5	7.90	7.78	211	0.36	11.5	11.6	15.4	12.1	12.7	15.2	-0.1	0.2	-0.8	23
6	9.38	9.24	251	0.33	15.4	14.8	15.5	15.5	14.2	16.3	-1.2	-0.01	-0.03	0.0

Table. The values of the masses M_{cl} in GeV, the mass term $\Delta E_{M.t.}$ (in MeV), the strangeness content C_S and the moments of inertia (in GeV^{-1}) for the $SU(3)$ projector configurations. The quantities for $B = 1$ hedgehog and the masses of $SU(2)$ embeddings $M_{cl}^{SU_2}$, in GeV are given for comparison. The components of the Wess-Zumino term WZ_3 and WZ_8 which are different from zero for the $SU(3)$ projector ansatz are also shown. ΔE in the last column is the quantum correction (in MeV) due to zero modes for the lowest energy state. The parameters of the model are $F_\pi = 186\,MeV$, $e = 4.12$.

As can be seen from this **Table**, the moments of inertia are pairwise equal, except for the case $B = 5$ when $\Theta_4 \neq \Theta_5$. These equalities are a consequence of the symmetry properties of our multiskyrmion configurations.

It should be noted that the energy of the $B = 4$ configuration is very close to that of the $B = 4$ toroidal soliton obtained in 1988 [12] within the axially symmetrical generalization of the ansatz by Balachandran et al. [13]. Within the accuracy of our calculations we cannot be certain which configuration has the lowest energy in the chirally symmetrical limit. For $B = 6$ the torus-like $SO(3)$ configuration has energy considerably higher than the energy of the $SU(3)$ projector ansatz presented in the **Table**.

Let us add here that the configuration of a double-torus form was considered also within the $SU(2)$-version of the model several years ago [14]. It has the energy somewhat higher than the energy of the known minimal energy configuration.

3. Following [10, 11, 15] we consider the contribution of the Wess-Zumino (WZ) term which defines the quantum numbers of the system in the quantization procedure. Its expression is given by (2) above. As usually, we introduce the time-dependent collective coor-

dinates for the quantization of zero modes according to the relation: $U(\vec{r},t) = A(t)U_0(\vec{r})A^\dagger(t)$. Next we perform some integration by parts and rewrite the expression for the WZ-term as:

$$L^{WZ} = \frac{-iN_c}{48\pi^2}\epsilon_{\alpha\beta\gamma}\int TrA^\dagger\dot{A}(R_\alpha R_\beta R_\gamma + L_\alpha L_\beta L_\gamma)d^3x, \tag{6}$$

where $L_\alpha = U_0^\dagger\partial_\alpha U_0 = iL_{k,\alpha}\lambda_k$ and $R_\alpha = \partial_\alpha U_0 U_0^\dagger = U_0 L_\alpha U_0^\dagger$, or

$$L^{WZ} = \frac{N_c}{24\pi^2}\int\sum_{k=1}^{k=8}\omega_k WZ_k d^3x = \sum_{k=1}^{k=8}\omega_k L_k^{WZ}, \tag{7}$$

with the angular velocities of rotation in the configuration space defined in the usual way: $A^\dagger\dot{A} = -\frac{i}{2}\omega_k\lambda_k$. Summation over repeated indices is assumed here and below. The functions WZ_k can be expressed through the chiral derivatives $\vec{L}_k$:

$$WZ_i = WZ_i^R + WZ_i^L = (R_{ik}(U_0) + \delta_{ik})WZ_k^L, \tag{8a}$$

$i,k = 1,...8$, and are given by [15]

$$\begin{aligned}
WZ_1^L &= -(L_1, L_4L_5 + L_6L_7) - (L_2L_3L_8)/\sqrt{3} - 2(L_8, L_4L_7 - L_5L_6)/\sqrt{3}\\
WZ_2^L &= -(L_2, L_4L_5 + L_6L_7) - (L_3L_1L_8)/\sqrt{3} - 2(L_8, L_4L_6 + L_5L_7)/\sqrt{3}\\
WZ_3^L &= -(L_3, L_4L_5 + L_6L_7) - (L_1L_2L_8)/\sqrt{3} - 2(L_8, L_4L_5 - L_6L_7)/\sqrt{3}\\
WZ_4^L &= -(L_4, L_1L_2 - L_6L_7) - (L_3L_5L_8)/\sqrt{3} + 2(\tilde{L}_8, L_1L_7 + L_2L_6)/\sqrt{3}\\
WZ_5^L &= -(L_5, L_1L_2 - L_6L_7) + (L_3L_4L_8)/\sqrt{3} - 2(\tilde{L}_8, L_1L_6 - L_2L_7)/\sqrt{3}\\
WZ_6^L &= (L_6, L_1L_2 + L_4L_5) + (L_3L_7L_8)/\sqrt{3} - 2(\tilde{\tilde{L}}_8, L_1L_5 - L_2L_4)/\sqrt{3}\\
WZ_7^L &= (L_7, L_1L_2 + L_4L_5) - (L_3L_6L_8)/\sqrt{3} + 2(\tilde{\tilde{L}}_8, L_1L_4 + L_2L_5)/\sqrt{3}\\
WZ_8^L &= -\sqrt{3}(L_1L_2L_3) + (L_8L_4L_5) + (L_8L_6L_7),
\end{aligned} \tag{9}$$

where (L_1, L_2L_3) denotes the mixed product of vectors $\vec{L}_1$, $\vec{L}_2$, $\vec{L}_3$, *ie* $(L_1, L_2L_3) = (\vec{L}_1\cdot\vec{L}_2\wedge\vec{L}_3)$ and $\tilde{L}_3 = (L_3 + \sqrt{3}L_8)/2$, $\tilde{L}_8 = (\sqrt{3}L_3 - L_8)/2$, $\tilde{\tilde{L}}_3 = (-L_3 + \sqrt{3}L_8)/2$, $\tilde{\tilde{L}}_8 = (\sqrt{3}L_3 + L_8)/2$ are the third and eighth components of the chiral derivatives in the (u,s) and (d,s) $SU(2)$-subgroups. $R_{ik}(U_0) = \frac{1}{2}Tr\lambda_i U_0\lambda_k U_0^\dagger$ is a real orthogonal matrix, and WZ_i^R are defined by the expressions (9) with the substitution $\vec{L}_k \to \vec{R}_k$. Relations similar to (9) can be obtained for $\widetilde{WZ}_3$ and $\widetilde{WZ}_8$; they are analogs of WZ_3 and WZ_8 for the (u,s) or (d,s) $SU(2)$ subgroups, thus clarifying the symmetry of the WZ-term in the different $SU(2)$ subgroups of $SU(3)$.

The baryon number of the $SU(3)$ skyrmions can be written also in terms of $\vec{L}_i$ in a form where its symmetry in the different $SU(2)$ subgroups of $SU(3)$ is more explicit:

$$B = -\frac{1}{2\pi^2}\int\left((L_1, L_2L_3) + (L_4, L_5\tilde{L}_3) + (L_6, L_7\tilde{\tilde{L}}_3) + \frac{1}{2}[(L_1, L_4L_7 - L_5L_6) + (L_2, L_4L_6 + L_5L_7)]\right)d^3r. \tag{10}$$

The contributions of the three $SU(2)$ subgroups enter the baryon number on an equal footing. In addition, mixed terms corresponding to the contribution of the chiral fields from different subgroups are also present.

The results of calculating the WZ-term according to (9) depend on the orientation of the soliton in the $SU(3)$ configuration space. The Guadagnini's quantization condition [10] was generalized in [15] for configurations of the "molecular" type to

$$Y_R^{min} = \frac{2}{\sqrt{3}}\frac{\partial L^{WZ}}{\partial\omega_8} \simeq \frac{N_cB(1-3C_S)}{3}, \tag{11}$$

where Y_R is the so-called right hypercharge characterizing the $SU(3)$ irrep under consideration, and the scalar strangeness content C_S is defined in terms of the real parts of the diagonal matrix elements of the matrix U:

$$C_S = \frac{< 1 - ReU_{33} >}{< 3 - Re(U_{11} + U_{22} + U_{33}) >}, \tag{12}$$

where $<>$ denotes the averaging or integration over the whole 3-dimensional space. When solitons are located in the (u,d) $SU(2)$ subgroup of $SU(3)$ only $\vec{L}_1$, $\vec{L}_2$ and $\vec{L}_3$ are different from zero, $C_S = 0$, WZ_8^R and WZ_8^L are both proportional to the B-number density, and the well known quantization condition [10] takes place

$$Y_R = N_c B/3. \tag{13}$$

The interpolation (11) does not work so well for configurations we consider here.

The expression for the rotation energy density of the system depending on the angular velocities of rotations in the $SU(3)$ collective coordinate space defined in Section 2 can be written in the following compact form [15]:

$$L_{rot} = \frac{F_\pi^2}{32}(\tilde{\omega}_1^2 + \tilde{\omega}_2^2 + .. + \tilde{\omega}_8^2) +$$

$$+ \frac{1}{16e^2}\Bigg\{(\vec{s}_{12} + \vec{s}_{45})^2 + (\vec{s}_{45} + \vec{s}_{67})^2 + (\vec{s}_{67} - \vec{s}_{12})^2 + \frac{1}{2}\Bigg((2\vec{s}_{13} - \vec{s}_{46} - \vec{s}_{57})^2 + (2\vec{s}_{23} + \vec{s}_{47} - \vec{s}_{56})^2 +$$

$$+(2\tilde{\vec{s}}_{34} + \vec{s}_{16} - \vec{s}_{27})^2 + (2\tilde{\vec{s}}_{35} + \vec{s}_{17} + \vec{s}_{26})^2 + (2\tilde{\vec{s}}_{36} + \vec{s}_{14} + \vec{s}_{25})^2 + (2\tilde{\vec{s}}_{37} + \vec{s}_{15} - \vec{s}_{24})^2\Bigg)\Bigg\}, \tag{14a}$$

or

$$L_{rot} = V_{ik}\tilde{\omega}_i\tilde{\omega}_k/2 = V_{ik}g_{li}g_{mk}\omega_l\omega_m/2 \tag{14b}$$

Here $\vec{s}_{ik} = \tilde{\omega}_i\vec{L}_k - \tilde{\omega}_k\vec{L}_i$, $i,k = 1,2...8$ are the $SU(3)$ indices, and $\tilde{\vec{s}}_{34} = (\vec{s}_{34} + \sqrt{3}\vec{s}_{84})/2$, $\tilde{\vec{s}}_{35} = (\vec{s}_{35} + \sqrt{3}\vec{s}_{85})/2$, $\tilde{\vec{s}}_{36} = (-\vec{s}_{36} + \sqrt{3}\vec{s}_{86})/2$, $\tilde{\vec{s}}_{37} = (-\vec{s}_{37} + \sqrt{3}\vec{s}_{87})/2$, *ie* similarly to the definitions of $\tilde{L}_3$ and $\tilde{L}_8$. g_{li} are given in (15). To get (14) we have used the identity: $\vec{s}_{ab}\vec{s}_{cd} - \vec{s}_{ad}\vec{s}_{cb} = \vec{s}_{ac}\vec{s}_{bd}$. The formula (14) possesses remarkable symmetry relative to the different $SU(2)$ subgroups of $SU(3)$. The functions L_8 or $\tilde{L}_8$ do not enter (14) nor the expression (10) for the baryon number density. The functions $\tilde{\omega}_i$ are connected to the body-fixed angular velocities of $SU(3)$ rotations by the transformations $\hat{\tilde{\omega}} = U_0^\dagger\hat{\omega}U_0 - \hat{\omega}$, or, equivalently

$$\tilde{\omega}_i = (R_{ik}(U_0^\dagger) - \delta_{ik})\omega_k = g_{ki}\omega_k. \tag{15}$$

Here $R_{ik}(V^\dagger) = R_{ki}(V)$ is a real orthogonal matrix, $i,k = 1,...8$, and $\tilde{\omega}_i^2 = 2(\omega_i^2 - R_{kl}(U_0)\omega_k\omega_l)$.

The expression for the static energy can be obtained from (14) by means of the substitution $\tilde{\omega}_i \to 2L_i$ and $\vec{s}_{ik} \to 2\vec{n_{ik}}$, [15] with $\vec{n}_{ik}$ being the cross product of $\vec{L}_i$ an $\vec{L}_k$ *ie* $\vec{n}_{ik} = \vec{L}_i \wedge \vec{L}_k$. From (14) we have then the inequality [15]

$$E_{stat} - M.t. \geq 3\pi^2 B(F_\pi/e), \tag{16}$$

which was obtained first by Skyrme [1] for the $SU(2)$ model.

Eight diagonal moments of inertia and 28 off-diagonal ones define the rotation energy, a quadratic form in $\omega_i\omega_k$ as follows from (14) and (15). The analytical expressions for the moments of inertia are too lengthy to be reproduced here. Fortunately, it is possible to perform calculations without explicit analytical formulas, by substituting (15) into (14).

For configurations generated by $SU(3)$ projectors the Lagrangian of the system can be written in terms of angular velocities of rotation and moments of inertia in the form (in the body-fixed system):

$$L_{rot} = \frac{\Theta_1}{2}(\omega_1^2+\omega_2^2)+\frac{\Theta_3}{2}\omega_3^2+\frac{\Theta_4}{2}\omega_4^2+\frac{\Theta_5}{2}\omega_5^2+\frac{\Theta_6}{2}(\omega_6^2+\omega_7^2)+\frac{\Theta_8}{2}\omega_8^2+\Theta_{38}\omega_3\omega_8+WZ_3\omega_3+WZ_8\omega_8. \quad (17)$$

After the standard quantization procedure the Hamiltonian of the system, $H=\omega_i\partial L/\partial\omega_i - L$, is a bilinear function of the generators R_i of $SU(3)$ rotations:

$$H = \frac{R_1^2+R_2^2}{2\Theta_1}+\Theta_8\frac{(R_3-WZ_3)^2}{2D_{38}}+\frac{R_4^2}{2\Theta_4}+\frac{R_5^2}{2\Theta_5}+\frac{R_6^2+R_7^2}{2\Theta_6}+$$
$$+\Theta_3\frac{(R_8-WZ_8)^2}{2D_{38}}-\frac{\Theta_{38}}{D_{38}}(R_3-WZ_3)(R_8-WZ_8), \quad (18)$$

where $D_{38}=\Theta_3\Theta_8-\Theta_{38}^2$. For the states belonging to a definite $SU(3)$ irrep the rotation energy can be written in terms of the second order Casimir operators of the $SU(2)$ and $SU(3)$ groups:

$$E_{rot} = \frac{N(N+1)-R_3^2}{2\Theta_1}+\frac{U(U+1)-R_{3,us}^2}{2\bar{\Theta}_4}+\frac{V(V+1)-R_{3,ds}^2}{2\Theta_6}+$$
$$+\frac{\Theta_8(R_3-WZ_3)^2}{2D_{38}}+\frac{\Theta_3(R_8-WZ_8)^2}{2D_{38}}-\frac{\Theta_{38}}{D_{38}}(R_3-WZ_3)(R_8-WZ_8)+\frac{(R_4^2-R_5^2)(\Theta_5-\Theta_4)}{4\Theta_4\Theta_5} \quad (19)$$

with $\bar{\Theta}_4 = 2\Theta_4\Theta_5/(\Theta_4+\Theta_5)$.

$$R_{3,us}=R_3/2+\sqrt{3}R_8/2,\ R_{8,us}=\sqrt{3}R_3/2-R_8/2,$$
$$R_{3,ds}=-R_3/2+\sqrt{3}R_8/2,\ R_{8,ds}=\sqrt{3}R_3/2+R_8/2, \quad (20)$$

with $R_3=R_{3,ud}$, $R_8=R_{8,ud}$; $R_{3,ab}$, $R_{8,ab}$ denote the 3^{rd} and 8^{th} generators of the (a,b) $SU(2)$ subgroup of $SU(3)$.

N, U and V are the values of the right isospin in the (u,d) $SU(2)$ subgroup, and so called U-spin and V-spin. They are connected with the second order Casimir operator of the $SU(3)$ group $C_2(SU_3)=\frac{1}{3}(p^2+q^2+pq)+p+q$; p,q being the numbers of the upper and low indices in the tensor describing the $SU(3)$ irrep (p,q) via the symmetric relation $C_2(SU_3)=N(N+1)+U(U+1)+V(V+1)-(R_3^2+R_8^2)/2$. The hypercharge $Y=2R_8/\sqrt{3}$. The terms linear in the angular velocities present in the Lagrangian due to the Wess-Zumino term cancel in the Hamiltonian.

For a state with the lowest energy which belongs to the $SU(3)$ singlet with $(p,q)=0$ the quantum correction simplifies to:

$$\Delta E = \frac{1}{2D_{38}}(\Theta_8 WZ_3^2+\Theta_3 WZ_8^2-2\Theta_{38}WZ_3WZ_8). \quad (21a)$$

The quantities D_{38}, $\Theta_3+\Theta_8$, $WZ_3^2+WZ_8^2$ do not depend on the choice of the particular subgroup of $SU(3)$ group, therefore $Eq.(21)$ also is invariant, as one expects from general arguments, since

$$\Theta_8 WZ_3^2+\Theta_3 WZ_8^2-2\Theta_{38}WZ_3WZ_8=(\Theta_8+\Theta_3)(WZ_3^2+WZ_8^2)-\Theta_3WZ_3^2-\Theta_8WZ_8^2-2\Theta_{38}WZ_3WZ_8.$$

The generalization of $(21a)$ to an arbitrary case is

$$\Delta E^{min}=\Theta_{ij}^{-1}WZ_iWZ_j/2, \quad (21b)$$

where Θ^{-1}_{ij} is the matrix inverse to the tensor of inertia Θ_{ij}. Eq. (21b) is valid for all cases except the degenerate case when $Det\Theta_{ij}=0$. Note, that the case of the $SU(2)$ embedding just corresponds to the degenerate case, with $\Theta_8=0$, and this leads to the rigorous quantization condition [10].

As can be seen from the **Table**, for the configurations we consider here and for our choice of the subgroups $\Theta_{38}^2 \ll \Theta_3\Theta_8$, so a small correction due to Θ_{38} can be neglected, and $\Delta E \simeq WZ_3^2/(2\Theta_3)+WZ_8^2/(2\Theta_8)$. Since the moments of inertia are of the order of magnitude $\sim 10\,GeV^{-1}$ and both WZ_3 and WZ_8 are not greater ~ 1, the quantum correction for the lowest states does not exceed several tens of MeV: see the **Table**.

The quantum correction from the $SU(3)$ zero modes which should be added to the classical energy of the soliton located originally in the (u,d) $SU(2)$ subgroup equals to $\Delta E = 3B/4\Theta_F \sim 365\,MeV$ [16].

Note, that for the $SO(3)$ solitons considered in [13, 11] the WZ-term is equal to zero, and for this reason the lowest quantized state was an $SU(3)$ singlet without any quantum correction to its mass.

4. The FSB part of the mass terms in the Lagrangian density which defines, in particular, the mass splittings inside $SU(3)$ multiplets is defined, as usual, by

$$L_{FSB}=\frac{F_D^2m_D^2-F_\pi^2m_\pi^2}{24}Tr(1-\sqrt{3}\lambda_8)(U+U^\dagger-2)+\frac{F_D^2-F_\pi^2}{48}Tr(1-\sqrt{3}\lambda_8)(Ul_\mu l^\mu+l_\mu l^\mu U^\dagger) \qquad (22)$$

Here m_D is the mass of K, D or B meson. The ratios F_D/F_π are known to be 1.22 and 1.7 ± 0.2 for, respectively, kaons and D-mesons. The L_{FSB} given by (22) is sufficient to describe the mass splittings of the octet and decuplet of baryons within the collective coordinate quantization approach [17].

The contribution to the classical mass of the solitons from the $SU(3)$ projector ansatz which comes from FSB part of the Lagrangian (22), considered as a perturbation, is not small even for strangeness:

$$\Delta M = 2C_S\Delta E_{M.t}\left(\frac{F_K^2m_K^2}{F_\pi^2m_\pi^2}-1\right) \qquad (23)$$

where $\Delta E_{M.t}$ is the FS mass term contribution shown in the **Table**. Numerically, it equals $1.13\,GeV$ for $B=2$ and $3.2\,GeV$ for $B=6$. Thus it makes sense to include the FSB part of the mass term into the minimization procedure.

We do not consider here the quantum corrections due to the rotations in the ordinary space because they are explicitily zero for the lowest states with even B $(J=0)$, or small [18].

Our investigations have shown that the space of local minima for the $SU(3)$ skyrmions has a rather complicated form. In addition to the known local minima - the $SU(2)$ embeddings, $SO(3)$ solitons and skyrmion molecules - there are also other local minima best approximated by the $SU(3)$ projector ansaetze [3, 4].

It is possible to quantize these configuratons and to estimate the spectrum of states by means of the collective coordinate quantization procedure. The total sum of the classical mass and of the zero modes quantum corrections can be the smallest for some baryon numbers. However, to draw the final conclusions one needs to calculate the Casimir energies of solitons which are essentially different for solitons of different form and size. This is a very complicated problem solved approximately [19] -[21] only for the $B=1$ hedgehog configurations. For the case of $SU(2)$ embeddings it was, however, possible to draw conclusions of physical relevance under the natural assumption that the unknown Casimir energy, or loop corrections, cancel in the difference of energies of states with different flavours, and that

the states with u, d flavours can be identified with ordinary nuclei [22]. But this assumption may be incorrect for the states generated by the $SU(3)$ projector ansatz.

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References

1. T.H.R. Skyrme, Proc. Roy. Soc., London, A260,127 (1961); Nucl.Phys. 31, 556 (1962)
2. C. Houghton, N. Manton, P.M. Sutcliffe, Nucl. Phys. B510, 507 (1998)
3. T. Ioannidou, B. Piette, W.J. Zakrzewski, J. Math. Phys. 40, 6353 (1999); Proc. of Internat. Conference "Mathematical methods in modern physics", Tbilisi, 1998, pp. 91-123, hep-th/9811071
4. T. Ioannidou, B. Piette, W.J. Zakrzewski, J. Math. Phys. 40, 6223 (1999); hep-th/9904160
5. E. Braaten, S. Townsend, L. Carson, Phys. Lett. B235,147 (1990)
6. R.A. Battye, P.M. Sutcliffe, Phys. Lett. B391, 150 (1997); Phys.Rev.Lett. 79, 363 (1997)
7. V.B. Kopeliovich, B. Schwesinger, B.E. Stern, JETP Lett. 62, 185 (1995)
8. E. Witten, Nucl. Phys. B223, 422; 433 (1983)
9. G.S. Adkins, C.R. Nappi, E. Witten, Nucl. Phys. B228,552 (1983); G.S. Adkins, C.R. Nappi, Nucl. Phys. B233, 109 (1984)
10. G. Guadagnini, Nucl. Phys. B236, 35 (1984)
11. R.L. Jaffe, C.L. Korpa, Nucl. Phys. B258, 468 (1985); A.P. Balachandran et al, Nucl. Phys. B256, 525 (1985)
12. A.I. Issinsky, V.B. Kopeliovich, B.E. Stern, Sov. J. Nucl. Phys. 48, 133 (1988)
13. A.P. Balachandran et al, Phys. Rev. Lett. 52, 887 (1984)
14. E. Sorace, M. Tarlini, Phys. Lett. 232B, 154 (1989)
15. V.B. Kopeliovich, JETP 85, 1060 (1997); Nucl. Phys. A639, 75c (1998); Phys. Atom. Nucl.56, 1084 (1993)
16. V.B. Kopeliovich, Phys. Lett. B259, 234 (1991); V.B. Kopeliovich, B. Schwesinger, B.E. Stern, Nucl. Phys. A549,485 (1992)
17. B. Schwesinger, H. Weigel, Phys. Lett. B267, 438 (1991); H. Weigel, Int. J. Mod. Phys. A11, 2419 (1996)
18. P. Irwin, hep-th/9804142; J.P. Garrahan, M. Schvellinger, N.N. Scoccola, hep-ph/9906432
19. B. Moussalam, Ann. of Phys. (N.Y.) 225, 264 (1993); F. Meier, H. Walliser, Phys. Rept. 289, 383 (1997)
20. H. Walliser, Phys. Lett. B432, 15 (1998)
21. N. Scoccola, H. Walliser, Phys. Rev. D58, 094037 (1998)
22. V.B. Kopeliovich, W.J. Zakrzewski, JETP Lett. 69, 721 (1999) [Pis'ma v ZhETF 69, 675 (1999)]; hep-ph/9909365